Kamnitzer

ref. Ginzburg : Perverse sheaves on a loop group

Mirkovic-Vilonen : Geometric Langlands duality

Frenkel : Recent progress in the Langlands group

G : reductive group $/\mathbb{C}$ $\quad SL_n, GL_n$

$K = \mathbb{C}((t)) = \mathbb{C}[t^{-1}, t]] \supset \mathcal{O} = \mathbb{C}[[t]]$

$Gr = G(K)/G(\mathcal{O})$ affine Grassmannian

ind. variety

$Gr_1 \subset Gr_2 \subset \cdots$ $\qquad Gr = \bigcup Gr_n = \varinjlim_n Gr_n$

proj. var. $\quad$ closed emb.

Ways to think about Gr

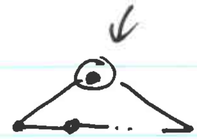
① It's a flag variety for $G(K)$

GL_n/P — parabolic $\quad$ partial flag var.

GL_n/B — Borel $\quad$ flag var.

$GL_n\Big/\begin{pmatrix} * & * \\ & * \end{pmatrix}{}^{n-k}_{k} = Gr(k, n)$

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c \oplus \mathbb{C}d$: affine Lie alg.



Then $G(\mathcal{O})$: maximal parabolic subgroup (distinguished)

corr. to $\mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}c \oplus \mathbb{C}d$

② $G/B \cong G_c/T_c$ $\qquad G_c$ = compact form of G

T_c = max. torus

$LG_c = \mathrm{Maps}(S^1, G_c)$

$Gr \xrightarrow[\text{deformation retract}]{\sim} LG_c/G_c \simeq \Omega G_c$ based loop group

(G_c : constant loops)

③ X : smooth alg. curve $/\mathbb{C}$ $\qquad X = \mathbb{P}^1, \mathbb{A}^1$

$x \in X$

$$Gr_{X,x} := \{ (P, \varphi) \quad P : \text{principal } G\text{-bdle on } X \}$$

$$\varphi : \underset{\substack{\uparrow \\ \text{trivial}}}{P_0}|_{X \setminus x} \xrightarrow{\sim} P_{X \setminus x}$$

<u>Th.</u> $Gr \cong Gr_{X,x}$

<u>Proof</u> (sketch) Let $(P, \varphi) \in Gr_{X,x}$. Pick a coord at x

pick a n.b.d U of x and a triv. ψ of P on U.

Then on $U \setminus x$

I have two trivializations

$$P_0|_{U \setminus x} \xrightarrow{\varphi} P|_{U \setminus x} \xrightarrow{\psi} P_0|_{U \setminus x}$$

get $f : U \setminus x \to G$

gives an element of $G(K)$

Changing ψ' right multiplies f by $f' \in G(\mathcal{O})$

$\therefore$ Get a well-defined elem. of Gr

$Gr_{X,x}$: moduli functor.

④ $Gr = G(K)/G(\mathcal{O})$ $\qquad G = GL_n$

$$GL_n(K)/GL(\mathcal{O}) \cong \{ \mathcal{O} \text{ lattices in } K^n \}$$

$$= \{ L \subset K^n \mid L : \text{free } \mathcal{O} \text{ submodule}, \ L \otimes_{\mathcal{O}} K = K^n \}$$

cf. $SL_2(\mathbb{R}) / SL_2(\mathbb{Z})$ lattices in $\mathbb{R}^2$

$GL_n(K) \curvearrowright$ right hand side

$GL_n(\mathcal{O}) = \mathrm{Stab}_{GL_n(K)}(\mathcal{O}^n)$

Or $[v_1 | v_2 | \cdots | v_n] \in G(K)$ $v_i \in K^n$

Consider $L = \mathrm{Span}_{\mathcal{O}}(v_1, \cdots, v_n) \subset K^n$

Let $\Lambda = \mathrm{Hom}(\mathbb{C}^\times, T)$ $T \subset G$ max lattice

coweight lattice

If $\mu \in \Lambda$ get $t^\mu \in Gr$ $\in \mathbb{Z}^n$

e.g. $G = SL_n$ $\Lambda = \{(\mu_1, \cdots, \mu_n) \mid \mu_1 + \cdots + \mu_n = 0\}$

$$t^\mu = \begin{bmatrix} t^{\mu_1} & & 0 \\ & \ddots & \\ 0 & & t^{\mu_n} \end{bmatrix}$$

$\Lambda \subset Gr$

Consider $G(\mathcal{O}) \curvearrowright Gr = G(K)/G(\mathcal{O})$ left multi.

analogous to B-orbits on G/B

$Gr^\lambda = G(\mathcal{O}) \cdot t^\lambda$ $\lambda \in \Lambda_+$: dominant coweight

Lemma

There is a unique orbit through each t^λ $\lambda \in \Lambda_+$

and these are all the orbits.

(proof for $G = GL_n$)

$\begin{bmatrix} & & \end{bmatrix}$ $\quad$ $\mathcal{O}$-row and column operations

get to t^λ for $\lambda \in \Lambda_+$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ //

(cf. $\Lambda_+ = W \backslash W_{\mathrm{aff}} / W$)

$\dim \mathrm{Gr}^\lambda = \langle 2\lambda, \rho \rangle$

smooth $\qquad$ half sum of pos. roots

usually not projective

$\overline{\mathrm{Gr}^\lambda}$ $\quad$ closure in Gr (but can take in the finite dim subvar. in Gr)

proj.

usually singular

$$\overline{\mathrm{Gr}^\lambda} = \bigcup_{\mu \leq \lambda} \mathrm{Gr}^\mu$$

$\overline{\mathrm{Gr}^\lambda}$ is smooth iff $\mathrm{Gr}^\lambda = \overline{\mathrm{Gr}^\lambda}$ iff λ is minuscule coweight

(NB. Gr^λ is a vector bdle over a flag variety

λ: regular $\Rightarrow$ full flag.

$\underline{\text{Ex.}}$ $G = GL_n$

$\lambda = (\underbrace{1,1,\cdots,1}_{k}, \underbrace{0 \cdots 0}_{n-k}) = \omega_k$

$$\overline{Gr^\lambda} = Gr^\lambda = G(\mathcal{O}) / \mathrm{Stab}_{G(\mathcal{O})} t^\lambda = G(\mathcal{O}) / \begin{array}{cc} & k \\ n-k & \begin{bmatrix} * & * \\ & * \end{bmatrix} \end{array}$$

these entries from $t\mathcal{O}$

$$\left(\begin{array}{l} g t^\lambda = t^\lambda h \quad h \in G(\mathcal{O}) \\ t^{-\lambda} g t^\lambda \in G(\mathcal{O}) \\ \begin{bmatrix} t & & & 0 \\ & \ddots & & \\ & & t \ 1 & \\ 0 & & & \ddots \ 1 \end{bmatrix} \end{array} \right.$$

$$= GL_n / \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} = Gr(k,n)$$ usual finite dim'l Grassmannian

---

Goal: $P_{G(\mathcal{O})} Gr$ $\quad$ $G(\mathcal{O})$-equiv. perverse sheaves

But before doing it, we talk about classical Satake isomorphism

G : split reductive $/ \mathbb{F}_q$ $\quad$ $G = GL_n$

$K = \mathbb{F}_q((t)) \supset \mathcal{O} = \mathbb{F}_q[[t]]$

or $\mathbb{Q}_p$ $\quad$ $\mathbb{Z}_p$

$G(\mathcal{O}) \curvearrowright G(K) \circlearrowleft G(\mathcal{O})$ $\quad$ (instead of $G(K)/G(\mathcal{O})$

$\mathcal{H}(G(K), G(\mathcal{O}))$: Hecke algebra

$G(\mathcal{O})$ biinvariant $\underbrace{\text{functions}}_{\text{compactly supported}}$ on $G(K)$

$\chi_\lambda = \chi(G(\mathcal{O}) t^\lambda G(\mathcal{O}))$: characteristic function of the orbit

$\mathcal{H}(G(K), G(\mathcal{O}))$ = finite linear combination of the χ_λ.
spherical Hecke algebra
mult. convolution

(usual Hecke algebra $B(\mathbb{F}_q)$-bi-invariant functions on $G(\mathbb{F}_q)$

$$H(G(K), G(\mathcal{O})) \underset{\text{Satake isom.}}{\cong} \mathbb{C}[\Lambda]^W$$

comm. of $H(G(K), G(\mathcal{O}))$ is surprising

recall G : reductive $/\mathbb{C}$ $\longrightarrow$ $(\Lambda \supset R, \Lambda^\vee \supset R^\vee)$

Λ: coweight lattice, R: coroots, $\Lambda^\vee$: weight lattice, $R^\vee$: roots

$G^\vee$ Langlands dual group $\longrightarrow$ weight lattice, root, coweight, coroots

weight lattice $\cup$ Λ_+

$G(\mathcal{O})$ orbits in Gr_G $\leftarrow\!\!\!\sim$ Λ_+ $\rightsquigarrow$ irr. reps of $G^\vee$

coweights   weights

$\therefore\ \mathbb{C}[\Lambda]^W \cong$ ring of characters of $G^\vee$

$\cong \operatorname{Rep} G^\vee$

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
$\vdots$	$\vdots$

geometric Satake correspondence

$\operatorname{Rep} G^\vee$: tensor category of f.d. reps of $G^\vee$

$\mathrel{\rotatebox{90}{$\cong$}}$

$\mathcal{P}_{G(\mathcal{O})} Gr$

tensor category of perverse sheaves constructible w.r.t. $G(\mathcal{O})$-orbits